TWO-BODY LATTICE HAMILTONIANS WITH FIRST AND SECOND NEAREST-NEIGHBORING-SITE INTERACTIONS*

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The IVth International School and Workshop on Few-Body Systems

30 September 2024, Khabarovsk

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*Based on a joint work with **S. Lakaev** and **S.Abdukhakimov** of Samarkand University.

Motivation and problem setup

In this presentation, our main object of study is a TWO-particle system on a TWO-dimensional (2D) lattice. But to make the introduction easier and softer, we start with an example of a one-particle system on a 1-dimensional lattice.

LATTICE SIMPLEST EXAMPLE: One-dimensional lattice $\Longleftrightarrow \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2 \ldots\}$, the set of entire numbers. Hilbert space:

$$l_2 = l_2(\mathbb{Z}) = \left\{ \widehat{f} : \mathbb{Z} \to \mathbb{C} \,\middle|\, \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 < \infty \right\}$$

Kinetic energy operator of a particle on the 1D-lattice $\mathbb{Z}$ is simply the second finite difference operator (up to a constant):

$$(\widehat{H}_0 f)(n) := -\frac{1}{2}\widehat{f}(n-1) - \frac{1}{2}\widehat{f}(n+1) + \widehat{f}(n).$$

Plus potential, say, a local operator

$$(\widehat{V}\widehat{f})(n) := \widehat{V}(n)\widehat{f}(n), \quad n \in \mathbb{Z}$$

where $\widehat{V}$ is a (decreasing as $|n| \to \infty$) real-valued function on $\mathbb{Z}$.

In (quasi)momentum space: Perform the Fourier transform $\mathcal{F} : l_2(\mathbb{Z}) \to L_2(\mathbb{T})$,

$$f(p) = (\mathcal{F}\widehat{f})(p) := \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \mathrm{e}^{ipn} \widehat{f}(n), \quad p \in [-\pi, \pi] =: \mathbb{T}, \quad f \in L_2(\mathbb{T}).$$

Then H_0 transforms into

$$\big(H_0 f\big)(p) = (1 - \cos p) f(p).$$

This implies that the spectrum of H_0 (and, hence, the spectrum of $\widehat{H}_0$) is purely absolutely continuous and fills the interval

$$\sigma(H_0) = [\min_{p\in\mathbb{T}}(1 - \cos p), \max_{p\in\mathbb{T}}(1 - \cos p)] = [0, 2].$$

A lot of things is known on the one-particle 1D-Hamiltonians (there is a scattering theory, see say, [Yafaev 2017]). Particular case of Jacobi matrices/operators. [Belyaev, Sandhas, AM 1997] used to explain enhancement of molecular-nuclear transitions due new-threshold resonances.

N-body ($N \geq 1$) problems on lattices: setup and a review in [Mattis 1986]. Including lattice dimensions up to 3 and even more. Since 1980s, a major contribution due to S.Lakaev and his students in Samarkand + coworkers from other cities/countries.

Interest to the few-body lattice problems is motivated, in paticular, by:

- Few-body lattice Hamiltonian may be viewed as a MINIMALIST version of the corresponding Bose- or Fermi-Hubbard model involving a fixed finite number of particles of a certain type.
- These hamiltonians represent a natural approximation for their continuous counterparts allowing to study few-body phenomena in the context of the theory of BOUNDED operators.
- The simplest and natural model for description of few-body systems formed by particles traveling through PERIODIC structures, say, for ulracold atoms injected into optical crystals created by the interference of counter-propagating laser beams.
- EFIMOV EFFECT, originally attributed to the three-body systems moving in $\mathbb{R}^3$ (1969/70). Efimov effect is present in three-body systems on the three-dimensional lattice $\mathbb{Z}^3$ [Lakaev:1993] $+$ [Albeverio Lakaev et al 2004, 2012].

Remark. The existence of Efimov-type phenomena:

- in a 5-boson system on a line $\mathbb{R}^1$ [Nishida et al 2010],
- in a 4-boson system on a plane $\mathbb{R}^2$ [Nishida 2017],
- for 3 spinless fermions moving on the plane $\mathbb{R}^2$ [Nishida et al 2013].

  In the latter case, a mathematical proof is available [Gridnev 2014], [Tamura 2019], and the phenomenon acquired the name of a super Efimov effect, because of the double exponential convergence of the binding energies to the three-body threshold.

One may guess that similar phenomena take place in the lattice case. Nothing has yet been done.

Now, introduce the Hamiltonian we discuss now.

Let $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ be the two-dimensional lattice and $\ell^{2,a}(\mathbb{Z}^2 \times \mathbb{Z}^2) \subset \ell^2(\mathbb{Z}^2 \times \mathbb{Z}^2)$, the Hilbert space of square–summable antisymmetric functions:

$$\widehat{f} \in \ell^{2,a}(\mathbb{Z}^2 \times \mathbb{Z}^2) \Longleftrightarrow \widehat{f}(y,x) = -\widehat{f}(x,y), \quad \forall x,y \in \mathbb{Z},$$
$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} |\widehat{f}(x,y)|^2 < \infty.$$

In the position-space, the Hamiltonian $\widehat{\mathbf{H}}_{\lambda\mu}$ for a system of two fermions with a first and second nearest-neighboring-site interaction potential $\widehat{\mathbf{V}}_{\lambda\mu}$ is an operator on $\ell^{2,a}(\mathbb{Z}^2 \times \mathbb{Z}^2)$ of the following form:

$$\widehat{\mathbf{H}}_{\lambda\mu} = \widehat{\mathbf{H}}_0 + \widehat{\mathbf{V}}_{\lambda\mu},\ \lambda,\mu \in \mathbb{R}. \tag{1}$$

Here, $\widehat{\mathbf{H}}_0$ is the kinetic energy operator of the system, defined on $\ell^{2,a}(\mathbb{Z}^2 \times \mathbb{Z}^2)$ as $(-1)\times$ finite-difference Laplacian (up to a constant) in 2D, i.e. on $\mathbb{Z}^2$:

$$[\widehat{\mathbf{H}}_0 \widehat{f}](x_1, x_2) = \sum_{s_1 \in \mathbb{Z}^2} \widehat{\varepsilon}(x_1 - s_1)\widehat{f}(s_1, x_2) + \sum_{s_2 \in \mathbb{Z}^2} \widehat{\varepsilon}(x_2 - s_2)\widehat{f}(x_1, s_2), \ \widehat{f} \in \ell^{2,a}(\mathbb{Z}^2 \times \mathbb{Z}^2), \quad (2)$$

where

$$\widehat{\varepsilon}(s) = \begin{cases} 2, & |s| = 0, \\ -\frac{1}{2}, & |s| = 1, \\ 0, & |s| > 1, \end{cases} \quad (3)$$

with $|s| = |s_1| + |s_2|$ for $s = (s_1, s_2) \in \mathbb{Z}^2$.

The first and second nearest-neighboring-site interaction potential $\widehat{\mathbf{V}}_{\lambda\mu}$ is the operator of multiplication by a function $\widehat{v}_{\lambda\mu}$,

$$[\widehat{\mathbf{V}}_{\lambda\mu}\widehat{f}](x_1,x_2)=\widehat{v}_{\lambda\mu}(x_1-x_2)\widehat{f}(x_1,x_2),\ \widehat{f}\in\ell^{2,a}(\mathbb{Z}^2\times\mathbb{Z}^2), \tag{4}$$

where

$$\widehat{v}_{\lambda\mu}(s)=\begin{cases}\frac{\lambda}{2}, & |s|=1,\\ \frac{\mu}{2}, & |s|=2,\\ 0, & s=0 \text{ or } |s|>2.\end{cases} \tag{5}$$

Notice that x_1 and x_2 are positions of the particles 1 and 2 on the lattice $\mathbb{Z}^2$.
All the three operators

$$\widehat{\mathbf{H}}_0,\ \widehat{\mathbf{V}}_{\lambda\mu}, \text{ and } \widehat{\mathbf{H}}_{\lambda\mu}=\widehat{\mathbf{H}}_0+\widehat{\mathbf{V}}_{\lambda\mu}$$

(for $\lambda,\mu\in\mathbb{R}$) are bounded and self-adjoint.

Let $\mathbb{T}^2\equiv[-\pi,\pi]\times[-\pi,\pi]$, and let $L^{2,a}(\mathbb{T}^2\times\mathbb{T}^2)$ be the Hilbert space of square-integrable antisymmetric functions on $\mathbb{T}^2\times\mathbb{T}^2$.

The quasimomentum-space version of the Hamiltonian $\widehat{\mathbf{H}}_{\lambda\mu} = \widehat{\mathbf{H}}_0 + \widehat{\mathbf{V}}_{\lambda\mu}$ reads as

$$\mathbf{H}_{\lambda\mu} := (\mathcal{F}\otimes\mathcal{F})\widehat{\mathbf{H}}_{\lambda\mu}(\mathcal{F}\otimes\mathcal{F})^*,$$

where $\mathcal{F}\otimes\mathcal{F}$ denotes the Fourier transform. The operator $\mathbf{H}_{\lambda\mu}$ acts on $L^{2,a}(\mathbb{T}^2\times\mathbb{T}^2)$ and has the form $\mathbf{H}_{\lambda\mu} = \mathbf{H}_0 + \mathbf{V}_{\lambda\mu}$, where $\mathbf{H}_0 = (\mathcal{F}\otimes\mathcal{F})\widehat{\mathbf{H}}_0(\mathcal{F}\otimes\mathcal{F})^*$ is the multiplication operator:

$$[\mathbf{H}_0 f](p,q) = [\varepsilon(p)+\varepsilon(q)]f(p,q),$$

with

$$\varepsilon(p) := \sum_{i=1}^{2}\big(1-\cos p_i\big), \quad p=(p_1,p_2)\in\mathbb{T}^2,$$

the *dispersion relation* of a single fermion. The interaction $\mathbf{V}_{\lambda\mu} = (\mathcal{F}\otimes\mathcal{F})\widehat{\mathbf{V}}_{\lambda\mu}(\mathcal{F}\otimes\mathcal{F})^*$ is the integral operator

$$[\mathbf{V}_{\lambda\mu}f](p,q) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} v_{\lambda\mu}(p-u)f(u,p+q-u)\mathrm{d}u$$

with the kernel function

$$v_{\lambda\mu}(p) = \lambda\sum_{i=1}^{2}\cos p_i + \mu\sum_{i=1}^{2}\cos 2p_i + 2\mu\sum_{i=1}^{2}\sum_{i\neq j=1}^{2}\cos p_i\cos p_j, \quad p=(p_1,p_2)\in\mathbb{T}^2.$$

The Floquet-Bloch decomposition of $\mathbf{H}_{\lambda\mu}$ and fiber Hamiltonians $H_{\lambda\mu}(K)$

Since $\widehat{\mathbf{H}}_{\lambda\mu}$ commutes with the representation of the discrete group $\mathbb{Z}^2$ by shift operators on the lattice, the space $L^{2,a}(\mathbb{T}^2\times\mathbb{T}^2)$ and $\mathbf{H}_{\lambda\mu}$ admit decomposition into the von Neumann direct integral:

$$L^{2,a}(\mathbb{T}^2\times\mathbb{T}^2)\simeq\int_{K\in\mathbb{T}^2}^{\oplus}L^{2,o}(\mathbb{T}^2)\,\mathrm{d}K \tag{6}$$

and

$$\mathbf{H}_{\lambda\mu}\simeq\int_{K\in\mathbb{T}^2}^{\oplus}H_{\lambda\mu}(K)\,\mathrm{d}K, \tag{7}$$

where $L^{2,o}(\mathbb{T}^2)$ is the Hilbert space of odd functions on $\mathbb{T}^2$. The fiber Hamiltonian $H_{\lambda\mu}(K)$, $K\in\mathbb{T}^2$, acting on $L^{2,o}(\mathbb{T}^2)$, is of the form

$$H_{\lambda\mu}(K):=H_0(K)+V_{\lambda\mu}, \tag{8}$$

where $H_0(K)$ is the operator of multiplication by the function

$$\mathcal{E}_K(p):=2\sum_{i=1}^{2}\Big(1-\cos\tfrac{K_i}{2}\cos p_i\Big) \tag{9}$$

and the perturbation operator $V_{\lambda\mu}$ is given by

$$[V_{\lambda\mu}f](s) = \frac{\lambda}{(2\pi)^2}\sum_{i=1}^{2}\sin s_i \int\limits_{\mathbb{T}^2} \sin t_i f(t)\mathrm{d}t + \frac{\mu}{(2\pi)^2}\sum_{i=1}^{2}\sin 2s_i \int\limits_{\mathbb{T}^2} \sin 2t_i f(t)\mathrm{d}t \tag{10}$$
$$+ \frac{\mu}{2\pi^2}\sum_{i=1}^{2}\sum_{j=1,\, j\neq i}^{2}\sin s_i \cos s_j \int\limits_{\mathbb{T}^2} \sin t_i \cos t_j f(t)\mathrm{d}t.$$

Obviously, both the operators $H_0(K)$ and $V_{\lambda\mu}$ are bounded and self-adjoint. The parameter $K \in \mathbb{T}^2$ is nothing but the *two-particle center-of-mass quasimomentum*

Moreover, $V_{\lambda\mu}$ is finite rank, $\dim \mathrm{Ran}(V_{\lambda\mu}) \leq 6$ for any $\lambda, \mu \in \mathbb{R}$.

The essential spectrum of the (fiber) two-body Hamiltonians

Depending on $\lambda, \mu \in \mathbb{R}$, the rank of $V_{\lambda\mu}$ varies but never exceeds six. Hence, by Weyl's theorem, for any $K \in \mathbb{T}^2$ the essential (continuous) spectrum $\sigma_{\mathrm{ess}}(H_{\lambda\mu}(K))$ of $H_{\lambda\mu}(K)$ coincides with the spectrum of $H_0(K)$, i.e.,

$$\sigma_{\mathrm{ess}}(H_{\lambda\mu}(K)) = \sigma(H_0(K)) = [\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)], \tag{11}$$

with

$$\mathcal{E}_{\min}(K) := \min_{p\in\mathbb{T}^2} \mathcal{E}_K(p) = 2\sum_{i=1}^{2}\Big(1-\cos\tfrac{K_i}{2}\Big) \geq \mathcal{E}_{\min}(0) = 0,$$
$$\mathcal{E}_{\max}(K) := \max_{p\in\mathbb{T}^2} \mathcal{E}_K(p) = 2\sum_{i=1}^{2}\Big(1+\cos\tfrac{K_i}{2}\Big) \leq \mathcal{E}_{\max}(0) = 8,$$

where

$$\mathcal{E}_K(p) := 2\sum_{i=1}^{2}\Big(1-\cos\tfrac{K_i}{2}\cos p_i\Big). \tag{12}$$

Main results

Theorem 1. *Suppose that, counting multiplicities. $H_{\lambda\mu}(0)$ has n eigenvalues below (resp. above) the essential spectrum for some $\lambda, \mu \in \mathbb{R}$. Then for each $K \in \mathbb{T}^2$ the operator $H_{\lambda\mu}(K)$ has at least n eigenvalues below (resp. above) its essential spectrum, counting multiplicities.*

Denote by $\mu_0^{\pm}$ and $\mu_1^{\pm}$ the following numbers:

$$\mu_0^{\pm} = \frac{88 - 30\pi \pm \sqrt{1044\pi^2 - 6720\pi + 10816}}{240\pi - 24\pi^2 - 512}\pi, \tag{13}$$

and

$$\mu_1^{\pm} = \frac{128 - 16\pi - 9\pi^2 \pm \sqrt{225\pi^4 - 1440\pi^3 + 3904\pi^2 - 10240\pi + 16384}}{120\pi - 12\pi^2 - 256}. \tag{14}$$

Note that the numerical values of $\mu_0^{\pm}$ and $\mu_1^{\pm}$ are as follows:

$$\mu_0^- = -5.6172..., \quad \mu_0^+ = -2.0623..., \quad \mu_1^- = -5.7523..., \quad \mu_1^+ = -2.9272...,$$

and, hence, these numbers satisfy the relations

$$\mu_1^- < \mu_0^- < \mu_1^+ < \mu_0^+ < 0. \tag{15}$$

By using the numbers μ_0^+, μ_0^- and μ_1^+, μ_1^- we introduce the following non-overlapping connected components of the (λ, μ) plane.

$$\mathcal{C}_0^- = \{(\lambda,\mu)\in\mathbb{R}^2 : \lambda > -\frac{8(\mu-\mu_0^+)(\mu-\mu_0^-)}{(\mu-\mu_1^+)(\mu-\mu_1^-)},\ \mu>\mu_1^+\},$$

$$\mathcal{C}_1^- = \{(\lambda,\mu)\in\mathbb{R}^2 :\ \lambda < -\frac{8(\mu-\mu_0^+)(\mu-\mu_0^-)}{(\mu-\mu_1^+)(\mu-\mu_1^-)},\ \mu>\mu_1^+\}$$
$$\cup\{(\lambda,\mu)\in\mathbb{R}^2 :\ \lambda\in\mathbb{R},\ \mu=\mu_1^+\}$$
$$\cup\{(\lambda,\mu)\in\mathbb{R}^2 :\ \lambda > -\frac{8(\mu-\mu_0^+)(\mu-\mu_0^-)}{(\mu-\mu_1^+)(\mu-\mu_1^-)},\ \mu_1^-<\mu<\mu_1^+\},$$

$$\mathcal{C}_2^- = \{(\lambda,\mu)\in\mathbb{R}^2 : \lambda < -\frac{8(\mu-\mu_0^+)(\mu-\mu_0^-)}{(\mu-\mu_1^+)(\mu-\mu_1^-)},\ \ \mu_1^-<\mu<\mu_1^+\}$$
$$\cup\{(\lambda,\mu)\in\mathbb{R}^2 :\ \lambda\in\mathbb{R},\ \mu=\mu_1^-\}$$
$$\cup\{(\lambda,\mu)\in\mathbb{R}^2 : \lambda > -\frac{8(\mu-\mu_0^+)(\mu-\mu_0^-)}{(\mu-\mu_1^+)(\mu-\mu_1^-)},\ \ \mu<\mu_1^-\},$$

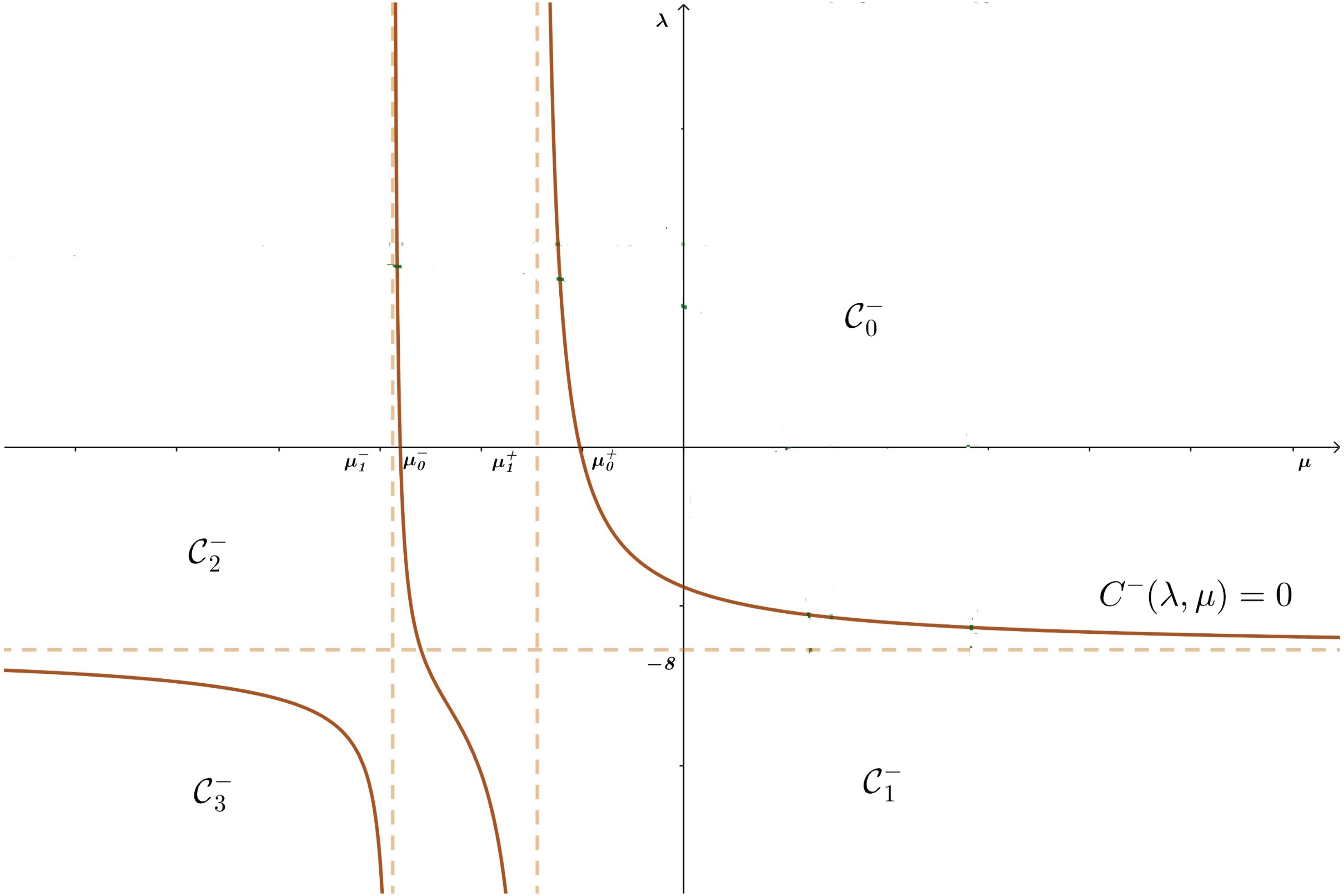
$$\mathcal{C}_3^- = \{(\lambda,\mu)\in\mathbb{R}^2 : \lambda < -\frac{8(\mu-\mu_0^+)(\mu-\mu_0^-)}{(\mu-\mu_1^+)(\mu-\mu_1^-)},\ \ \mu<\mu_1^-\}$$

and

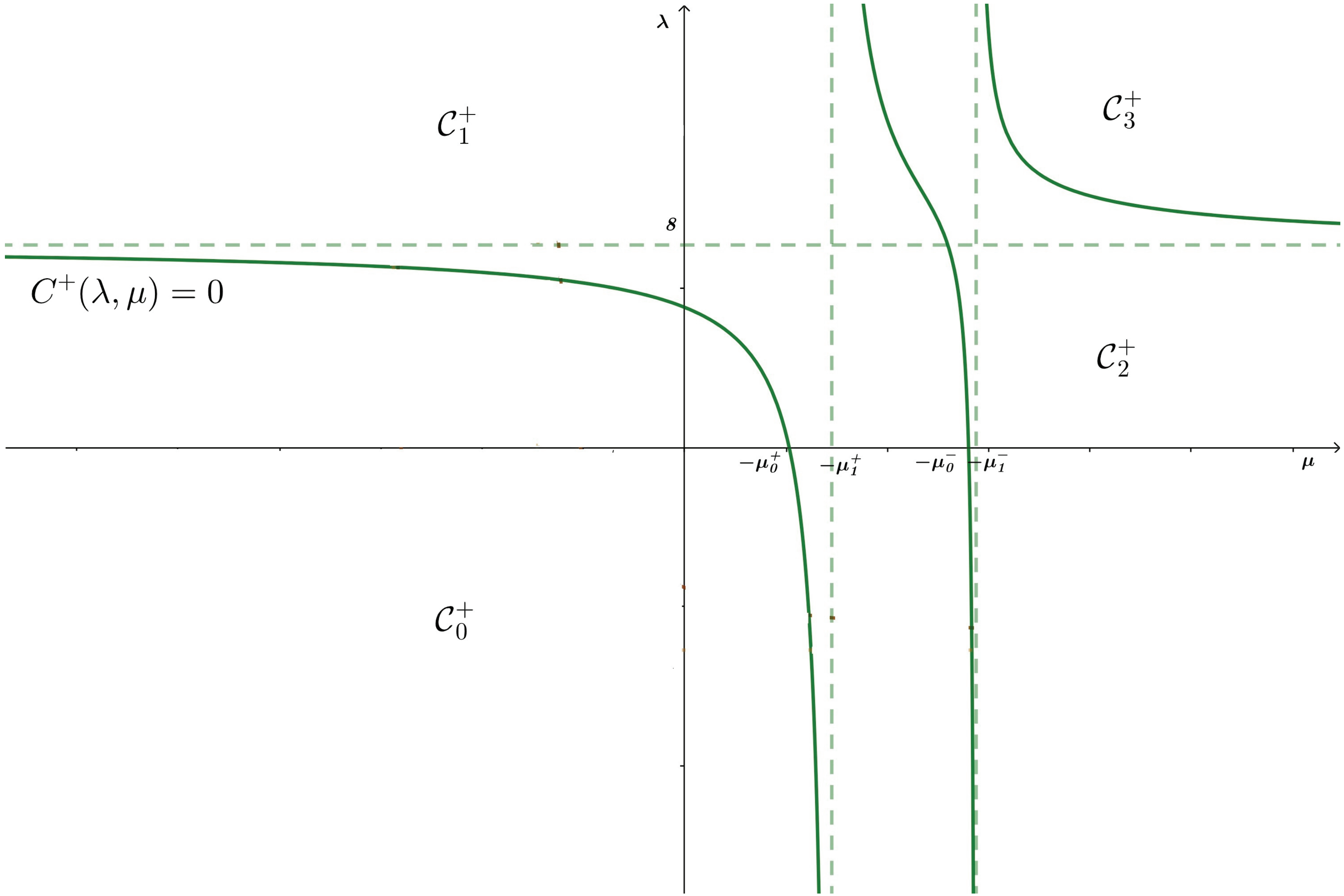
$$
\begin{aligned}
\mathcal{C}_0^+ &= \{(\lambda,\mu)\in\mathbb{R}^2 : \lambda < \frac{8(\mu+\mu_0^+)(\mu+\mu_0^-)}{(\mu+\mu_1^+)(\mu+\mu_1^-)},\ \mu < -\mu_1^+\},\\
\mathcal{C}_1^+ &= \{(\lambda,\mu)\in\mathbb{R}^2 :\ \lambda > \frac{8(\mu+\mu_0^+)(\mu+\mu_0^-)}{(\mu+\mu_1^+)(\mu+\mu_1^-)},\ \mu < -\mu_1^+\}\\
&\qquad \cup\{(\lambda,\mu)\in\mathbb{R}^2 :\ \lambda\in\mathbb{R},\ \mu = -\mu_1^+\}\\
&\qquad \cup\{(\lambda,\mu)\in\mathbb{R}^2 :\ \lambda < \frac{8(\mu+\mu_0^+)(\mu+\mu_0^-)}{(\mu+\mu_1^+)(\mu+\mu_1^-)},\ -\mu_1^+ < \mu < -\mu_1^-\},\\
\mathcal{C}_2^+ &= \{(\lambda,\mu)\in\mathbb{R}^2 :\ \lambda > \frac{8(\mu+\mu_0^+)(\mu+\mu_0^-)}{(\mu+\mu_1^+)(\mu+\mu_1^-)},\ -\mu_1^+ < \mu < -\mu_1^-\}\\
&\qquad \cup\{(\lambda,\mu)\in\mathbb{R}^2 :\ \lambda\in\mathbb{R},\ \mu = -\mu_1^-\}\\
&\qquad \cup\{(\lambda,\mu)\in\mathbb{R}^2 : \lambda < \frac{8(\mu+\mu_0^+)(\mu+\mu_0^-)}{(\mu+\mu_1^+)(\mu+\mu_1^-)},\ \mu > -\mu_1^-\},\\
\mathcal{C}_3^+ &= \{(\lambda,\mu)\in\mathbb{R}^2 : \lambda > \frac{8(\mu+\mu_0^+)(\mu+\mu_0^-)}{(\mu+\mu_1^+)(\mu+\mu_1^-)},\ \mu > -\mu_1^-\}.
\end{aligned}
$$

It turns out that in each of the above components $\mathcal{C}_k^-$, the number of eigenvalues of the operator $H_{\lambda\mu}(0)$, lying below its essential spectrum, remains constant. In a similar way, any of the components $\mathcal{C}_k^+$ is a domain where the number of eigenvalues of $H_{\lambda\mu}(0)$, lying above the essential spectrum (11), does not vary. Both these facts are established in the following theorem.

Theorem 2. *Let $\mathcal{C}^-$ be one of the above connected components $\mathcal{C}_k^-$, $k = 0,1,2,3$, of the partition of the (λ,μ)-plane. Then for any $(\lambda,\mu) \in \mathcal{C}^-$ the number $n_-(H_{\lambda\mu}(0))$ of eigenvalues of $H_{\lambda\mu}(0)$ (counting multiplicities) lying below the essential spectrum $\sigma_{\rm ess}\big(H_{\lambda\mu}(0)\big)$ remains constant. Analogously, let $\mathcal{C}^+$ be one of the above connected components $\mathcal{C}_k^+$, $k = 0,1,2,3$, of the partition of the (λ,μ)-plane. Then for any $(\lambda,\mu) \in \mathcal{C}^+$ the number $n_+(H_{\lambda\mu}(0))$ of eigenvalues of $H_{\lambda\mu}(0)$ (counting multiplicities) lying above $\sigma_{\rm ess}\big(H_{\lambda\mu}(0)\big)$ remains constant.*



Domains $\mathcal{C}_n^-$ with fixed number of eigenvalues (equal to $2n$) below the essential (continuous) spectrum of $H_{\lambda\mu}(0)$.



Domains $\mathcal{C}_n^+$ with fixed number of eigenvalues (equal to $2n$) below the essential (continuous) spectrum of $H_{\lambda\mu}(0)$.

The result below concerns the number of eigenvalues of the fiber Hamiltonian $H_{\lambda\mu}(K)$ for various K and (λ,μ).

Theorem 3. *Let $K\in\mathbb{T}^2$ and $(\lambda,\mu)\in\mathbb{R}^2$. Then for the numbers $n_+(H_{\lambda\mu}(K))$ and $n_-(H_{\lambda\mu}(K))$ of eigenvalues of the operator $H_{\lambda\mu}(K)$ lying, respectively, above and below its essential spectrum $\sigma_{\rm ess}\big(H_{\lambda\mu}(K)\big)$, the following two series of implications hold:*

$$\begin{array}{lcl} (\lambda,\mu)\in\mathcal{C}_3^+\cap\mathcal{C}_0^- & \Longrightarrow & n_+(H_{\lambda\mu}(K))=6,\\ (\lambda,\mu)\in\mathcal{C}_2^+\cap\mathcal{C}_0^- \,\text{or}\, (\lambda,\mu)\in\mathcal{C}_2^+\cap\mathcal{C}_1^- & \Longrightarrow & n_+(H_{\lambda\mu}(K))\geq 4,\\ (\lambda,\mu)\in\mathcal{C}_1^+\cap\mathcal{C}_0^- \,\text{or}\, (\lambda,\mu)\in\mathcal{C}_1^+\cap\mathcal{C}_1^- & \Longrightarrow & n_+(H_{\lambda\mu}(K))\geq 2,\\ (\lambda,\mu)\in\overline{\mathcal{C}_0^+} & \Longrightarrow & n_+(H_{\lambda\mu}(K))\geq 0, \end{array} \tag{16}$$

and

$$\begin{array}{lcl} (\lambda,\mu)\in\mathcal{C}_3^-\cap\mathcal{C}_0^+ & \Longrightarrow & n_-(H_{\lambda\mu}(K))=6,\\ (\lambda,\mu)\in\mathcal{C}_2^-\cap\mathcal{C}_0^+ \,\text{or}\, (\lambda,\mu)\in\mathcal{C}_2^-\cap\mathcal{C}_1^+ & \Longrightarrow & n_-(H_{\lambda\mu}(K))\geq 4,\\ (\lambda,\mu)\in\mathcal{C}_1^-\cap\mathcal{C}_0^+ \,\text{or}\, (\lambda,\mu)\in\mathcal{C}_1^-\cap\mathcal{C}_1^+ & \Longrightarrow & n_-(H_{\lambda\mu}(K))\geq 2,\\ (\lambda,\mu)\in\overline{\mathcal{C}_0^-} & \Longrightarrow & n_-(H_{\lambda\mu}(K))\geq 0, \end{array} \tag{17}$$

where $\overline{\mathcal{A}}$ is the closure of the set $\mathcal{A}$.

In fact, the estimates for the numbers $n_+(H_{\lambda\mu}(K))$ and $n_-(H_{\lambda\mu}(K))$ of eigenvalues of the operator $H_{\lambda\mu}(K)$ obtained in Theorem 3 are sharp.

The next theorem establishes the exact number of eigenvalues of $H_{\lambda\mu}(0)$ outside its essential spectrum.

Theorem 4. *For various $\lambda,\mu\in\mathbb{R}$, the numbers and multiplicities of eigenvalues of $H_{\lambda\mu}(0)$ outside the set $\sigma_{\rm ess}\big(H_{\lambda\mu}(0)\big)$ are described in the following statements.*

(i) *For any $(\lambda,\mu)\in\mathcal{C}_{30}=\mathcal{C}_3^-$ the operator $H_{\lambda\mu}(0)$ has exactly three eigenvalues $z_1(\lambda,\mu;0)$, $z_2(\lambda,\mu;0)$ and $z_3(\lambda,\mu;0)$ of multiplicity two satisfying*

$$z_1(\lambda,\mu;0)<z_2(\lambda,\mu;0)<z_3(\lambda,\mu;0)<0. \tag{18}$$

(ii) *For any $(\lambda,\mu)\in\mathcal{C}_{20}:=\mathcal{C}_2^-\cap\mathcal{C}_0^+$ the operator $H_{\lambda\mu}(0)$ has two eigenvalues $z_1(\lambda,\mu;0)$ and $z_2(\lambda,\mu;0)$ of multiplicity two satisfying*

$$z_1(\lambda,\mu;0)<z_2(\lambda,\mu;0)<0 \tag{19}$$

and it has no eigenvalues in $(8,+\infty)$

(iii) *For any $(\lambda,\mu)\in\mathcal{C}_{21}:=\mathcal{C}_2^-\cap\mathcal{C}_1^+$, the operator $H_{\lambda\mu}(0)$ has two eigenvalues $z_1(\lambda,\mu;0)$ and $z_2(\lambda,\mu;0)$ of multiplicity two in $(-\infty,0)$ and it has one eigenvalue of multiplicity two in $(8,+\infty)$.*

(iv) *For any $(\lambda,\mu)\in\mathcal{C}_{11}:=\mathcal{C}_1^-\cap\mathcal{C}_1^+$, the operator $H_{\lambda\mu}(0)$ has two eigenvalues $z_1(\lambda,\mu)<0$ and $z_2(\lambda,\mu)>8$ of multiplicity two .*

(v) *For any* $(\lambda,\mu)\in \mathcal{C}_{10}:=\mathcal{C}_1^-\cap\mathcal{C}_0^+$, *the operator* $H_{\lambda\mu}(0)$ *has one eigenvalue* $z(\lambda,\mu;0)$ *of multiplicity two in* $(-\infty,0)$, *nevertheless it has no eigenvalues in* $(8,+\infty)$.

(vi) *For any* $(\lambda,\mu)\in \mathcal{C}_{00}:=\mathcal{C}_0^-\cap\mathcal{C}_0^+$, *the operator* $H_{\lambda\mu}(0)$ *has no eigenvalues outside of the essential spectrum.*

(vii) *For any* $(\lambda,\mu)\in \mathcal{C}_{01}:=\mathcal{C}_0^-\cap\mathcal{C}_1^+$, *the operator* $H_{\lambda\mu}(0)$ *has one eigenvalue* $z(\lambda,\mu;0)$ *of multiplicity two in* $(8,+\infty)$ *and it has no eigenvalues in* $(-\infty,0)$.

(ix) *For any* $(\lambda,\mu)\in \mathcal{C}_{02}:=\mathcal{C}_0^-\cap\mathcal{C}_2^+$, *the operator* $H_{\lambda\mu}(0)$ *has two eigenvalues* $z_1(\lambda,\mu;0)$ *and* $z_2(\lambda,\mu;0)$ *of multiplicity two satisfying*

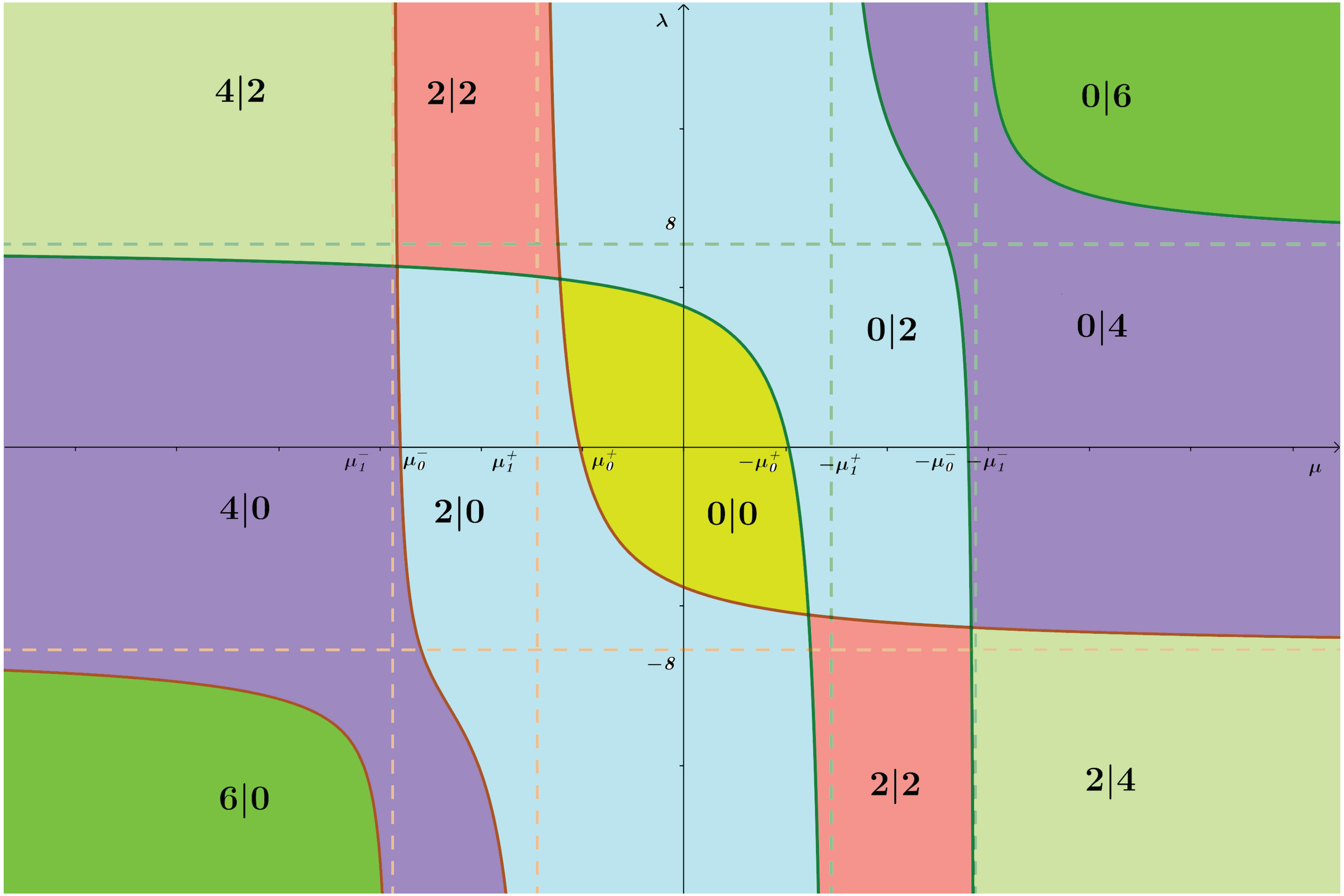
$$8<z_2(\lambda,\mu;0)<z_1(\lambda,\mu;0) \tag{20}$$

and it has no eigenvalues in $(-\infty,0)$.

(viii) *For any* $(\lambda,\mu)\in \mathcal{C}_{12}=\mathcal{C}_1^-\cap\mathcal{C}_2^+$, *the operator* $H_{\lambda\mu}(0)$ *has one eigenvalue of multiplicity two in* $(-\infty,0)$ *and it has two eigenvalues* $z_1(\lambda,\mu;0)$ *and* $z_2(\lambda,\mu;0)$ *of multiplicity two in* $(8,+\infty)$.

(x) *For any* $(\lambda,\mu)\in \mathcal{C}_{03}:=\mathcal{C}_3^+$, *the operator* $H_{\lambda\mu}(0)$ *has exactly three eigenvalues* $z_1(\lambda,\mu;0)$, $z_2(\lambda,\mu;0)$ *and* $z_3(\lambda,\mu;0)$ *of multiplicity two satisfying*

$$8<z_3(\lambda,\mu;0)<z_2(\lambda,\mu;0)<z_1(\lambda,\mu;0). \tag{21}$$



Partition of the (λ,μ)-plane of parameters $\lambda,\mu\in\mathbb{R}$ in the connected components $\mathcal{C}_{\alpha\beta},\alpha,\beta=0,1,2,3$. These components are tagged by the symbols $N_-|N_+$ formed of the numbers $N_-:=n_-(H_{\lambda,\mu}(0))$ and $N_+:=n_+(H_{\lambda,\mu}(0))$ of eigenvalues of $H_{\lambda\mu}(0)$ lying below and above the essential spectrum, resp.

Reference:

S.N. Lakaev, A.K. Motovilov, and S.Kh.Abdukhakimov, *Two-fermion lattice Hamiltonian with first and second nearest-neighboring-site interactions*, J. Phys. A: Math. Theor. **56** (2023), 315202 [23 pages];arXiv:2303.10491.